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LETTER TO THE EDITOR

Coupled-channel scattering and separation of coupled differential equations by generalized Darboux transformations

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Abstract. General methods for separation of coupled differential equations are applied in coupled-channel scattering theory with appropriate asymptotic conditions. It is shown that the problems with thresholds cannot be separated. Finally it is pointed out that separation of the coupled-channel s-wave problem without thresholds can be realized by the use of generalized Darboux transformations.

Recently there have been extensive studies of separation of coupled systems of differential equations (Humi 1985, Cao 1992 and references therein, Cannata and Ioffe 1992). Cannata and Ioffe (1992) had specifically the aim of discussing coupled-channel scattering theory. Here we want to generalize the results obtained previously, we point out the connections between Humi (1985), Cao (1992) and Cannata and Ioffe (1992) and discuss the difference between the cases with thresholds (Amado *et al* 1988a) (also referred to as non-resonant (Cao 1992)) and the ones without thresholds (Amado *et al* 1988b, Cannata and Ioffe 1992) (also referred to as resonant (Humi 1985)). The distinction between these two cases stems from the necessity of imposing proper boundary conditions at infinity and can be formulated in a rather general framework which allows for higher-order (higher-derivative) Darboux transformations (Infeld and Hull 1951, Amado *et al* 1990, Andrianov *et al* 1993).

For simplicity we shall restrict ourselves to two coupled channels and describe the radial s-wave equations corresponding to a quantum mechanical off-diagonal Hamiltonian:

$$\mathcal{H}^{\mathrm{OD}}\phi_E(r)\equiv(-\partial^2\sigma_0+\sigma_0\cdot n_0(r)+\sigma_1\cdot n_1(r)+\sigma_3\cdot n_3(r))\phi_E(r)=E\cdot\phi_E(r) \qquad (1)$$

where ϕ_E is a two component column, σ (the Pauli matrices) are defined as:

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

and $n(r)$ are real functions ($0\leq r<\infty$). The term like $\sigma_2\cdot n_2(r)$, with

$$\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

is excluded if we require (Hermiticity) that n_2 is real and the potential matrix should be real. We shall assume that $n_1\to0$ for $r\to\infty$. *A priori* we allow n_0, n_3 to approach constant values c_0 and c_3 at infinity. If in particular c_0, $c_3\to0$ we obtain the no-threshold (resonant) case.

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The solution to the problem will be searched for in terms of solutions of a system of equations which is separated (decoupled)

$$\mathscr{H}^{\mathrm{D}}\psi_E(r) \equiv (-\partial^2\sigma_0 + \sigma_0 \cdot m_0(r) + \sigma_3 \cdot m_3(r))\psi_E(r) = E\psi_E(r) \tag{2}$$

where the equivalence of equation (1) and equation (2) is given by the fact that, with exception of a restricted number of states, we require

$$Q\psi_E = \phi_E. \tag{3}$$

Q is a differential operator (Anderson and Camporesi 1990, Anderson 1991) like:

$$Q = \sum_n A_n(r)\partial^n$$

where $A_n(r)$ are matrix-valued functions and the sum can be truncated to $n = 0, 1, 2$, etc, giving thereby rise to various matrix differential operators already discussed in the literature.

The equivalence of $\mathscr{H}^{\mathrm{OD}}$ and $\mathscr{H}^{\mathrm{D}}$ can also be expressed by the well known intertwining relations

$$Q\mathscr{H}^{\mathrm{D}} = \mathscr{H}^{\mathrm{OD}}Q. \tag{4}$$

In the context of scattering theory $\mathscr{H}^{\mathrm{OD}}$ and $\mathscr{H}^{\mathrm{D}}$ are identical at infinity ($n_1 \to 0$ for $r \to \infty$). Then the intertwining relation (4) at infinity implies that *in presence of thresholds* (see, however, the remark later on)

$$Q_\infty = \alpha(r, \partial)\sigma_0 + \beta(r, \partial)\sigma_3.$$

When there are no thresholds there is no such restriction on Q_∞.

A physical interpretation of this algebraic result can easily be obtained by considering the action of Q_∞ on asymptotic states (Amado *et al* 1988a):

$$\psi_{1,E}^\infty \sim \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k_1 r} + S_1^{\mathrm{D}}\,\mathrm{e}^{\mathrm{i}k_1 r} \\ 0 \end{pmatrix} \qquad \psi_{2,E}^\infty \sim \begin{pmatrix} 0 \\ \mathrm{e}^{-\mathrm{i}k_2 r} + S_2^{\mathrm{D}}\,\mathrm{e}^{\mathrm{i}k_2 r} \end{pmatrix}$$

with $E = k_1^2 + c_0 + c_3 = k_2^2 + c_0 - c_3$.

If we require

$$Q_\infty \psi_E^\infty = \phi_E^\infty$$

with

$$\phi_E^\infty = \phi_{1,E}^\infty \sim \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k_1 r} + S_{11}^{\mathrm{OD}}\,\mathrm{e}^{\mathrm{i}k_1 r} \\ S_{12}^{\mathrm{OD}}\,\mathrm{e}^{\mathrm{i}k_2 r} \end{pmatrix}$$

or

$$\phi_E^\infty = \phi_{2,E}^\infty \sim \begin{pmatrix} S_{21}^{\mathrm{OD}}\,\mathrm{e}^{\mathrm{i}k_1 r} \\ \mathrm{e}^{-\mathrm{i}k_2 r} + S_{22}^{\mathrm{OD}}\,\mathrm{e}^{\mathrm{i}k_2 r} \end{pmatrix}$$

it is clear that Q_∞ has to be diagonal in the presence of thresholds ($k_1 \neq k_2$) in order to preserve the correct structure of the exponentials.

The importance of the previous result is made apparent by the fact that if we search in general (irrespectively of thresholds) for the linear combination of ψ_1^∞ and ψ_2^∞ which leads (when acted by Q_∞) to ϕ_1^∞ or ϕ_2^∞ we find that a solution with off diagonal S^{OD} is possible only for Q_∞ off-diagonal (see, e.g. equation (14) of Cannata and Ioffe 1992). We thus conclude that *the problem of coupled channels with thresholds cannot be separated and from now on we will restrict ourselves to problems without thresholds* (*resonant case*).

The problem of separation of coupled-channel scattering amounts to finding $\mathscr{H}^{\mathrm{D}}$ and Q starting from a given $\mathscr{H}^{\mathrm{OD}}$. The search for Q has been the subject of previous investigations (Humi 1985, Cao 1992) which, however, did not focus attention on the appropriate asymptotic conditions relevant for scattering. From the previous considerations it is clear that for the scattering problem only an off-diagonal Q_∞ can be accepted.

While the previous considerations have not made use of any explicit form for Q from now on we will discuss explicit realizations given in the literature. We only briefly mention the case which can be diagonalized by a constant matrix A ($n_0(r)\neq 0$; $n_1(r)\neq 0$, $n_3(r)=0$) (London 1932).

The classical expression for Q is the well known Darboux transformation:

$$Q(r,\partial)=A(r)+B\partial \tag{5}$$

where $A(r)$ and B (independent of r) are in general 2×2 matrices.

Humi (1985) has studied the case $B=\sigma_0$ and Cannata and Ioffe (1992) $B=\sigma_1+\sigma_3$. The connection between these two treatments is obtained by multiplying Q by a constant unitary matrix T (Andrianov *et al* 1992). The intertwining relations (4) yield:

$$TQ\mathscr{H}^{\mathrm{D}}=T\mathscr{H}^{\mathrm{OD}}T^{-1}TQ.$$

Since the operator $T\mathscr{H}^{\mathrm{OD}}T^{-1}$ can be interpreted as the operator $\mathscr{H}^{\mathrm{OD}}$ after a change of basis we can interpret TQ to be the new operator which satisfies equations (3) and (4). It is clear that if the transformation matrix is taken as $T=\frac{1}{2}(\sigma_1+\sigma_3)$ we realize the equivalence between the two choices provided TQ_∞ is still off-diagonal.

In explicit realizations the matrix Q of equation (5) depends on the functions $n(r)$ of equation (1).

In particular in Cannata and Ioffe (1992) the parametrization of A (equation (5)) dictated by the form of $B=\sigma_1+\sigma_3$ and by the requirement of separation is

$$A=\begin{pmatrix} W & -W+\varphi+\varphi'/\varphi \\ W-\varphi & W-\varphi'/\varphi \end{pmatrix}$$

where W and φ are arbitrary functions to be expressed in terms of n_0, n_1, n_3 of equation (1).

One can easily obtain

$$\varphi=\tfrac{2}{3}\int^r n_3(y)\,\mathrm{d}y$$

$$W=\frac{1}{\varphi(r)}\left[c+\tfrac{1}{2}(\varphi^2+\varphi')+\int^r (n_1(y)\varphi(y))\,\mathrm{d}y\right]$$

with c an arbitrary constant. The necessary compatibility condition in the sense discussed by Humi (1985) and Cao (1992) is

$$2n_0=2W^2+\varphi^2+\frac{\varphi''}{\varphi}-2W\left(\varphi+\frac{\varphi'}{\varphi}\right).$$

Next in order of complexity is the ansatz:

$$\begin{aligned}Q^{\mathrm{II}}(r,\partial)&=A(r)+D(r)\partial+\partial^2\\ &=a_0(r)\cdot\sigma_0+\boldsymbol{a}(r)\cdot\boldsymbol{\sigma}+b_0(r)\cdot\sigma_0\partial+\boldsymbol{b}(r)\cdot\boldsymbol{\sigma}\partial+\partial^2.\end{aligned} \tag{6}$$

A reparametrization of (6) can be obtained by the factorization:

$$Q^{\mathrm{II}}(V,W)=(\partial+V(r))(\partial+W(r)) \tag{7}$$

where $V(r)$ and $W(r)$ are arbitrary matrix-valued functions. The connection between the two parametrizations is given in terms of a matrix Riccati equation:

$$(W-\tfrac{1}{2}D)^2-(W-\tfrac{1}{2}D)'-\tfrac{1}{4}D^2-\tfrac{1}{2}D'+A=0$$
$$V=D-W.$$

We would like to stress that in order to avoid extensions which are trivial for our separation purposes we have to show that such Q^{II} cannot, in general, be reproduced multiplying a first-order differential operator like in equation (5) by a similar diagonal operator. We therefore study the conditions for such trivial extension $Q^{\mathrm{II}}_{\mathrm{trivial}}$;

$$Q^{\mathrm{II}}_{\mathrm{trivial}}=(\partial+V(r))(\partial+W^{\mathrm{D}}(r)) \tag{8}$$

with

$$W^{\mathrm{D}}(r)=w_0(r)\cdot\sigma_0+w_3(r)\cdot\sigma_3$$

and

$$V(r)=v_0(r)\cdot\sigma_0+\boldsymbol{v}(r)\cdot\boldsymbol{\sigma}.$$

The equality of (6) and (8) leads to the following conditions:

$$(v_0\pm v_3)^2+(v_0\pm v_3)'-(v_0\pm v_3)(b_0\pm b_3)-(b_0\pm b_3)'+(a_0\pm a_3)=0 \tag{9}$$

where v_0 and v_3 should be written as solutions of the linear equations:

$$\begin{pmatrix} b_1 & \mathrm{i}b_2 \\ b_2 & -\mathrm{i}b_1 \end{pmatrix}\begin{pmatrix} v_0 \\ v_3 \end{pmatrix}=\begin{pmatrix} b_0b_1+\mathrm{i}b_3b_2-a_1 \\ b_0b_2-\mathrm{i}b_1b_3-a_2 \end{pmatrix}$$

and v_1 and v_2 can be expressed in terms of these functions. Equation (9), as a constraint imposed on the parameters a and b, expresses the condition for triviality.

The results support the view that the ansatz (6) is in general not a trivial extension of (5). The generalization of our algebraic results to the $N\times N$ case seems rather straightforward.

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